Three paths to the rank metric

Finite geometry

M. Ceria
Politecnico di Bari

Finite projective spaces

Let us consider a prime power $q = p^h$ and denote by $\mathbb{F}_q$ the finite field with size q.

Finite projective space $\mathrm{PG}(n, q)$ (a.k.a. $\mathrm{PG}(V)$):

let $V := V(n+1, q)$ be a $n+1$-dimensional vector space over $\mathbb{F}_q$.
For $0 \leq k \leq n-1$ all k-dimensional projective subspaces of $\mathrm{PG}(n, q)$ are the $(k+1)$-vector subspaces of V.

- 0-projective subspace: *point* $\rightarrow$ 1- vector subspace;
- 1-projective subspace: *line* $\rightarrow$ 2- vector subspace;
- 2-projective subspace: *plane* $\rightarrow$ 3- vector subspace;
- 3-projective subspace: *solid* $\rightarrow$ 4- vector subspace;
- and so on.

Collineations, semilinear maps, projectivities

Collineation

If V, W are equidimensional $\mathbb{F}_q$ vector spaces, a **collineation** between $\mathrm{PG}(V)$ and $\mathrm{PG}(W)$ is a bijection between their points with the property of preserving incidence.

Let $\sigma \in Aut(\mathbb{F}_q)$ and $A \in GL_{n+1}(q)$

Semilinear isomorphism

$$F_{\sigma,A} : V(n+1,q) \to V(n+1,q)$$

such that, for $\overline{\mathbf{x}} = (x_1, ..., x_n)$:

$$F_{\sigma,A}(\overline{\mathbf{x}}) = \sigma(\overline{\mathbf{x}})A_T$$

where $\sigma(\overline{\mathbf{x}}) = (\sigma(x_1), ..., \sigma(x_n))$

$F_{\sigma,A} \in \Gamma L(n+1,q)$

Semilinear isomorphism

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$$F_{\sigma,A}(\bar{\mathbf{x}}) = \sigma(\bar{\mathbf{x}})A_T$$

where $\sigma(\bar{\mathbf{x}}) = (\sigma(x_1), ..., \sigma(x_n))$

It defines a bijection among the points of $\mathrm{PG}(n,q)$ which preserves incidence: **projective semilinear map**.

SEMILINEAR ISOMORPHISM

$$F_{\sigma,A} : V(n+1,q) \to V(n+1,q)$$

such that, for $\overline{\mathbf{x}} = (x_1, ..., x_n)$:

$$F_{\sigma,A}(\overline{\mathbf{x}}) = \sigma(\overline{\mathbf{x}})A_T$$

where $\sigma(\overline{\mathbf{x}}) = (\sigma(x_1), ..., \sigma(x_n))$

For $\sigma = Id$ we have a **projectivity**

Collineations, semilinear maps, projectivities

For $\sigma = Id$ we have a **projectivity**

$$P\Gamma L(n+1,q) = \{\text{ all projective semilinear maps}\}$$
$$PGL(n+1,q) = \{\text{ all projectivities}\}$$

Each automorphism $\sigma \in Aut(\mathbb{F}_q)$ induces a collineation of $\mathrm{PG}(n, q)$

$F_{\sigma,A}, G_{\sigma',B}$ induce the same collineation if and only if they differ by the multipilcation by a scalar matrix:

$$\sigma = \sigma' \quad A = \mu B, \, \mu \in \mathbb{F}_q^*$$

Collineations, semilinear maps, projectivities

$F_{\sigma,A}, G_{\sigma',B}$ induce the same collineation if and only if they differ by the multipilcation by a scalar matrix:

$$\sigma = \sigma' \quad A = \mu B, \, \mu \in \mathbb{F}_q^*$$

,

$$P\Gamma L(n+1,q) = \Gamma L_{n+1}(q)/Z(\Gamma L_{n+1}(q))$$
$$PGL(n+1,q) = GL_{n+1}(q)/Z(GL_{n+1}(q))$$

where

$$Z(\Gamma L_{n+1}(q)) = Z(GL_{n+1}(q)) = \{\lambda I : \lambda \in \mathbb{F}_q^*\} \simeq \mathbb{F}_q^*$$

Dual of $\mathrm{PG}(V)$

$\mathrm{PG}(n,q)^* := \mathrm{PG}(V^*)$: points are hyperplanes of $\mathrm{PG}(n,q)$ and so on; incidence is reversed.

Reciprocity - polarity

Reciprocity: a collineation ρ between $\mathrm{PG}(n,q)$ and $\mathrm{PG}(n,q)^*$.
If it has order 2 it is a **polarity**.

Textbook

ABNR

Gianira N. Alfarano, Martino Borello, Alessandro Neri, Alberto Ravagnani

Linear cutting blocking sets and minimal codes in the rank metric

Journal of Combinatorial Theory, Series A 192 (2022): 105658.

Support – VRMC

$C \leq (\mathbb{F}_{q^m})^n$ VRMC; $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$:

$$supp(\overline{\mathbf{v}}) = colsp(\Gamma(\overline{\mathbf{v}})) \leq (\mathbb{F}_q)^n$$

is the (rank) **support** of $\overline{\mathbf{v}} \in C$, and it is a $\mathbb{F}_q$-linear space.

Rank weight: $r(\overline{\mathbf{v}}) = dim_{\mathbb{F}_q}(supp(\overline{\mathbf{v}}))$

$D \leq C \leq (\mathbb{F}_{q^m})^n$

$$supp(D) = \sum_{v \in D} supp(\overline{\mathbf{v}}) \leq (\mathbb{F}_q)^n.$$

Non-degenerate code

$C \leq (\mathbb{F}_{q^m})^n$ VRMC with length n and dimension k.

It is **nondegenerate** if

$$\mathrm{Supp}(C) = \mathbb{F}_{q^n}$$

Effettive length

$$\dim(\mathrm{Supp}(C))$$

Rank-nondegenerate codes

$C \leq (\mathbb{F}_{q^m})^n$ VRMC

TFAE

- C rank-nondegenerate;
- for every $A \in GL_n(q)$, the code CA is nondegenerate w.r.t. the Hamming metric;
- The $\mathbb{F}_q$-span of the columns of any generator matrix of G has dimension n over $\mathbb{F}_q$;
- $d(C^\perp) \geq 2$.

We can isometrically embed a degenerate code in a smaller-length space.
For C nondegenerate, it holds $n \leq km$ (JP).

PROJECTIVE SYSTEMS (q-ANALOGUE)

R

$[n, k, d]_{q^m/q}$ **system**

$$U \leq (\mathbb{F}_{q^m})^k$$

$\mathbb{F}_q$-space, $\dim_{\mathbb{F}_q}(U) = n$ such that

$$\langle U \rangle_{\mathbb{F}_{q^m}} = (\mathbb{F}_{q^m})^k$$

and

$$d = n - \max\{\dim_{\mathbb{F}_q}(U \cap H) : H \leq (\mathbb{F}_{q^m})^k, \mathbb{F}_{q^m} - \text{hyperplane}\}$$

I.E

$$d = \min\{\dim_{\mathbb{F}_q}(U + H) : H \leq (\mathbb{F}_{q^m})^k, \mathbb{F}_{q^m} - \text{hyperplane}\} - m(k-1)$$

Equivalent projective systems

$U, V, [n, k, d]_{q^m/q}$ systems

Equivalent
if there is a $\mathbb{F}_{q^m}$-isomorphism $\phi : (\mathbb{F}_{q^m})^k \to (\mathbb{F}_{q^m})^k$ sending U to V

Standard equation

U, $[n,k,d]_{q^m/q}$ system.

Call Λ_r the set of all r-dimensional subspaces of $(\mathbb{F}_{q^m})^k$ over $\mathbb{F}_{q^m}$

$$\sum_{H\in\Lambda_r} |H \cap (U \setminus \{0\})| = (q^n - 1)\begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_{q^m}$$

Codes and systems

Set of equivalence classes of $[n, k, d]_{q^m/q}$ nondegenerate codes: $\mathcal{C}[n, k, d]_{q^m/q}$

Set of equivalence classes of $[n, k, d]_{q^m/q}$ systems: $\mathcal{U}[n, k, d]_{q^m/q}$

From codes to systems

We define a map

$$\Phi : \mathcal{C}[n,k,d]_{q^m/q} \to \mathcal{U}[n,k,d]_{q^m/q}$$

in this way:

- take $[C] \in \mathcal{C}[n,k,d]_{q^m/q}$
- let G be a generator matrix for C
- $\Phi([C])$: equivalence class of the $\mathbb{F}_q$-span of the columns of G.

From systems to codes

We define a map

$$\Psi : U[n,k,d]_{q^m/q} \to C[n,k,d]_{q^m/q}$$

in this way:

- take $[U] \in U[n,k,d]_{q^m/q}$
- fix a basis $\{g_1, ..., g_n\}$ for U;
- let G be the matrix whose columns are $g_1, ..., g_n$
- $\Phi([U])$: equivalence class of the code generated by G.

Recall from the basics

For $\overline{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$, $\overline{\mathbf{v}} = (v_1, \ldots, v_n)$, $v_i \in \mathbb{F}_{q^m}$, $1 \leq i \leq n$.

$$r(\overline{\mathbf{v}}) = \dim_{\mathbb{F}_q}(\langle v_1, \ldots, v_n \rangle)$$

$C \leq (\mathbb{F}_{q^m})^n$

- $C \neq 0$:

$$d_{min}(C) = \min\{r(\overline{\mathbf{v}}) : \overline{\mathbf{v}} \in C, \overline{\mathbf{v}} \neq \overline{\mathbf{0}}\}$$

- $C = 0$:

$$d_{min}(C) = d_{min}(0) = n + 1.$$

Recall from the basics

Note that

$$d_{min}(C) \leq d^H(C).$$

For $\overline{\mathbf{v}} \in (\mathbb{F}_{q^m})^n$, $\overline{\mathbf{v}} = (v_1, \ldots, v_n)$, $v_i \in \mathbb{F}_{q^m}$, $1 \leq i \leq n$.

$$r(\overline{\mathbf{v}}) = \min\{w^H(\overline{\mathbf{v}}A) : A \in GL_n(q)\}$$

Bases and dimensions

Take a finite-dimensional vector space V over $\mathbb{F}_q$.
Let $U, W \leq V$;

$$\mathcal{B} := \{\text{bases of } U\}$$

$$\max\{|B \cap W| : \quad B \in \mathcal{B}\} = \dim(U \cap W).$$

Again on the rank

C $[n,k]_{q^m/q}$ nondegenerate code, with generator matrix G. Let $\overline{\mathbf{u}} \in (\mathbb{F}_{q^m})^k$ a nonzero vector.

If U is the $[n,k]_{q^m/q}$ system generated by the columns of G over $\mathbb{F}_q$

$$r(\overline{\mathbf{u}}G) = n - \dim_{\mathbb{F}_q}(U \cap \langle \overline{\mathbf{u}} \rangle^{\perp}).$$

Back and forth

R

Φ, Ψ are well-defined maps and they're one the inverse of the other.

There's then a bijection between $C[n, k, d]_{q^m/q}$ and $U[n, k, d]_{q^m/q}$

Minimum distance

C $[n, k, d]_{q^m/q}$ code:

$$d \geq \dim_{\mathbb{F}_q}(supp(C)) - m(k-1)$$

Maximum rank

C $[n,k]_{q^m/q}$ nondegenerate code:

$$maxr(C) = \min\{n, m\}$$

Ravagnani
C $[n,k]_{q^m/q}$ code; $maxr(C) = k$. If $m \geq n$ then C has a basis given by vectors with all components in $\mathbb{F}_q$.

Generalized weights

$C \leq (\mathbb{F}_{q^m})^n$ VRMC.

v5 - Randrianarisoa

$$w_i(C) = \min\{\dim(A) : A \leq (\mathbb{F}_{q^m})^n, \text{ Frobenius closed } \dim_{\mathbb{F}_{q^m}}(C \cap A) \geq i\}$$

for $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C) = k$.

Randrianarisoa

C: $[n,k,d]_{q^m/q}$ nondegenerate code and
U: $[n,k,d]_{q^m/q}$ system associated to the code.

For any $i = 1, ..., \dim_{\mathbb{F}_{q^m}}(C) = k$

$$w_i(C) = n - \max\{\dim_{\mathbb{F}_q}(U \cap H) : H \leq (\mathbb{F}_{q^m})^k, \mathbb{F}_{q^m} - subspace, codim(H) = i\}$$

$$= \min\{\dim_{\mathbb{F}_q}(U + H) : H \leq (\mathbb{F}_{q^m})^k, \mathbb{F}_{q^m} - subspace, codim(H) = i\} - m(k-i)$$

SIMPLEX RMC

C $[mk, k]_{q^m/q}$ code with $k \geq 2$ and generator matrix G.

TFAE

- C nondegenerate;
- $colsp_{\mathbb{F}_q}(G) = (\mathbb{F}_{q^m})^k$;
- C 1-weight code, $d_{min}(C) = m$;
- $d_{min}(C^\perp) > 1$;
- $d_{min}(C^\perp) = 2$;
- C linearly equivalent to a code with generator matrix

$$G' = (I_k|\alpha I_k|...|\alpha^{m-1} I_k)$$

$\alpha \in \mathbb{F}_{q^m}$ with $\mathbb{F}_{q^m} = \mathbb{F}_q(\alpha)$.

1-WEIGHT CODES

C $[n, k, d]_{q^m/q}$ one-weight code with $k \geq 2$.

- **effective length**: km;
- $d = m$.

Isometry:

$$[km, k, m]_{q^m/q} \text{ simplex code.}$$

Linear set

Lunardon
U: $[n,k]_{q^m/q}$ system.

$\mathbb{F}_q$-linear set in $\mathrm{PG}(k-1,q)$ of **rank** n associated to U:

$$L_U = \{\langle \overline{\mathbf{u}} \rangle_{\mathbb{F}_{q^m}} : \overline{\mathbf{u}} \in U \setminus \{\overline{\mathbf{0}}\}\}$$

($\langle \overline{\mathbf{u}} \rangle_{\mathbb{F}_{q^m}}$ projective point corresponding to $\overline{\mathbf{u}}$).

Linear set

$V \leq (\mathbb{F}_{q^m})^k$, $\mathbb{F}_{q^m}$-subspace.

$$\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^m})$$

Weight of Λ in L_U

$$w_U(\Lambda) = \dim_{\mathbb{F}_q}(U \cap V).$$

Scattered linear sets

Blokhuis-Lavrauw

U: $[n, k]_{q^m/q}$ system.

$$|L_U| \leq \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

Scattered

When equality $\Leftrightarrow w_U(P) = 1$ for all $P \in L_U$.

Maximum: biggest possible rank.

Link with the Hamming metric

ABNR

U: $[n, k]_{q^m/q}$ system.

$$\sum_{P \in \mathbf{PG}(k-1, q^m)} \frac{q^{w_U(P)} - 1}{q - 1} = \frac{q^n - 1}{q - 1}.$$

Link with the Hamming metric

Sheekey - ABNR

U: $[n,k]_{q^m/q}$ system.

$P \in \mathrm{PG}(k-1, q^m)$

$$\boldsymbol{m}_U(P) := \frac{q^{w_U(P)} - 1}{q-1}$$

$$\sum_{P \in \mathbf{PG}(k-1, q^m)} m_U(P) = \frac{q^n - 1}{q-1}.$$

$U(n,k)_{q^m/q}$: $[n,k]_{q^m/q}$ systems
$P(n,k)_{q^m}$: $[n,k]_{q^m}$ projective systems

$$U(n,k)_{q^m/q} \to P((q^n-1)/(q-1),k)_{q^m}$$
$$U \mapsto (L_U, m_U)$$

Multiset L_U, m_U multiplicity function.

The map is compatible with the equivalence classes of these objects.

Projective systems and linear systems

$U[n,k]_{q^m/q}$: equivalence classes of $[n,k]_{q^m/q}$ systems
$P[n,k]_{q^m}$: equivalence classes of $[n,k]_{q^m}$ projective systems

$$Ext^H : U[n,k]_{q^m/q} \to P[(q^n-1)/(q-1),k]_{q^m}$$

Associated Hamming Code

$$Ext^H : U[n,k,d]_{q^m/q} \to P[(q^n-1)/(q-1), k, (q^n-q^{n-d})/(q-1)]_{q^m}$$

C nondegenerate $[n,k,d]_{q^m/q}$ VRMC.
Associated Hamming Code: any code in $(\Psi^H \circ Ext^H \circ \Phi)([C])$.
Parameters:

$$[(q^n-1)/(q-1), k, (q^n-q^{n-d})/(q-1)]_{q^m}$$

Associated Hamming Code

C nondegenerate $[n, k, d]_{q^m/q}$ VRMC; rank-weight distribution $\{A_i(C)\}$;

$$A_i^H(C^H) = \begin{cases} A_i(C) & \textbf{if } j = \frac{q^n - q^{n-i}}{q-1} \\ 0 & \textbf{otherwise} \end{cases}$$

Associated Hamming Code

C nondegenerate $[n, k, d]_{q^m/q}$ VRMC; generalized weights $\{w_i(C)\}$;

$$w_i^H(C^H) = \frac{q^n - q^{n-w_i(C)}}{q-1}$$

Minimal codeword

C: $[n, k, d]_{q^m/q}$ VRMC;

Minimal

$\overline{\mathbf{c}} \in C$: if there is $\overline{\mathbf{c}}'$ with $supp(\overline{\mathbf{c}}') \subseteq supp(\overline{\mathbf{c}})$ it means that the two codewords are one multiple of the other.

Linear cutting blocking set

U $[n,k]_{q^m/q}$ system is a **linear cutting blocking set** if for each H $\mathbb{F}_{q^m}$–hyperplane $\langle H \cap U\rangle_{\mathbb{F}_{q^m}} = H$.

Idea: the associated linear set L_U cutting blocking set in $\mathrm{PG}(k-1, q^m)$.

U: $[n,k]_{q^m/q}$ system. Linear cutting blocking set if and only if for each H, H' $\mathbb{F}_{q^m}$–hyperplanes in $(\mathbb{F}_{q^m})^k$

$$H \cap U \subseteq H' \cap U \Rightarrow H = H'$$

U $[n,k]_{q^m/q}$ linear cutting blocking set, for each H $\mathbb{F}_{q^m}$–hyperplane in $(\mathbb{F}_{q^m})^k$

$$|H \cap U| \geq q^{k-1}$$

A correspondence...

C $[n,k]_{q^m/q}$ nondegenerate code and U its associated system. Let G be a generator matrix for C.
$\overline{\mathbf{u}}, \overline{\mathbf{v}} \in (\mathbb{F}_{q^m})^k \setminus \{\overline{\mathbf{0}}\}$

$$supp(\overline{\mathbf{u}}G) \subseteq supp(\overline{\mathbf{v}}G) \Leftrightarrow (U \cap \langle \overline{\mathbf{u}} \rangle^{\perp}) \supseteq (U \cap \langle \overline{\mathbf{v}} \rangle^{\perp})$$

… THAT WE ALREADY SAW…

$$\Phi : C[n,k,d]_{q^m/q} \to U[n,k,d]_{q^m/q}$$
$$\Psi : U[n,k,d]_{q^m/q} \to C[n,k,d]_{q^m/q}$$

Φ, Ψ are well-defined maps and they're one the inverse of the other.

There's then a bijection between $C[n,k,d]_{q^m/q}$ and $U[n,k,d]_{q^m/q}$

Φ, Ψ are well-defined maps and they're one the inverse of the other.

They induce a bijection between minimal RMC and linear cutting blocking sets.

New minimal codes

C $[n, k]_{q^m/q}$ minimal code; G generator matrix, $\bar{\mathbf{u}} \in (\mathbb{F}_{q^m})^k$

The $[n+1, k]_{q^m/q}$ code $\overline{C}$ whose generator matrix is $(G|\bar{\mathbf{u}}_T)$ is minimal.

New minimal codes

$C\ [n,k]_{q^m/q}$ minimal code.

Then

$$\forall \overline{\mathbf{c}} \in C \quad r(\overline{\mathbf{c}}) \le \dim_{\mathbb{F}_q}(supp(C)) - k + 1$$

$$maxr(C) \le \dim_{\mathbb{F}_q}(supp(C)) - k + 1 \le n - k + 1$$

$k \geq 2$

$$n \geq k + m - 1$$

Connecting with Hamming minimal codes

C $[n,k]_{q^m/q}$ code.

$$\text{Hamming minimal} \Rightarrow \text{rank-minimal}$$
$$\not\Leftarrow$$

C $[n,k]_{q^m/q}$ nondegenerate code.

$$C \text{ rank-minimal} \Leftrightarrow C^H \text{ Hamming minimal}$$

Minimality condition

C $[n, k]_{q^m/q}$ code.

C is minimal if and only if, for each $\overline{\mathbf{c}}, \overline{\mathbf{c}}' \in C$ linearly independent, it holds

$$\sum_{\lambda \in \mathbb{F}_{q^m} \setminus \{0\}} q^{-r(\overline{\mathbf{c}} + \lambda \overline{\mathbf{c}}')} \neq (q^m - 1) q^{-r(\overline{\mathbf{c}})} - q^{-r(\overline{\mathbf{c}}')} + 1$$

Ashikmin-Barg condition

Hamming case

C $[n, k]_{q^m}$ code.

$$C \text{ minimal} \Leftrightarrow \frac{w_{\max}}{w_{\min}} < \frac{q^m}{q^m - 1}$$

In the rank-metric

The condition becomes trivial.

Some minimal codes

$C \quad [km, k, m]_{q^m/q}$ simplex code.

$\Rightarrow C$ is minimal

A C nondegenerate $[n, k]_{q^m/q}$ code with $n \geq (k-1)m + 1$ is minimal

$k = 3$

A C nondegenerate $[n, 3]_{q^m/q}$ code with $n \geq m + 2$ is minimal and with U as associated $[n, 3]_{q^m/q}$ system .

$$L_U \text{ scattered} \Rightarrow C \text{ minimal}$$

BLOKHUIS–LAVRAUW
If U $[n, k]_{q^m/q}$ system with L_U scattered then

$$n \leq \frac{km}{2}$$

Maximum scattered linear sets: equality.

Scattered linear sets

km even number - Csajbók-Marino-Polverino-Zullo

There's a system U with parameters $[km/2, k]_{q^m/q}$ s.t. L_U scattered

km odd number

Still much to do

Blockhuis-Lavrauw

k, m positive integers, q prime power. The there exist a $[ab, k]_{q^m/q}$ system s.t. L_U scattered every time $a \mid k$, $GCD(a, m) = 1$

$$ab < \begin{cases} \frac{km-m+3}{2} & q = 2,\ a = 1 \\ \frac{km-m+3+a}{2} & \text{otherwise} \end{cases}$$

Puncturing

U with parameters $[n, k]_{q^m/q}$ s.t. L_U scattered. If $n > k$ there is a $[n-1, k]_{q^m/q}$ system $V \subseteq U$ such that also L_V is scattered.

What happens then in dimension 3

$m \neq 3, 5 \mod 6$, $m \geq 4$.

There is then a nondegenerate minimal $[m+2, 3]_{q^m/q}$ code.

Existence of minimal codes

m, n, k positive integers, $n \geq k \geq 2$. If the value

$$\frac{(q^{mn}-1)(q^{m(n-1)}-1)}{(q^{mk}-1)(q^{m(k-1)}-1)}-\frac{1}{2}\sum_{i=2}^{m}\frac{1}{q^m-1}\begin{bmatrix} m \\ i \end{bmatrix}_q\prod_{j=0}^{i-1}(q^n-q^j)\left(\frac{q^{mi}-1}{q^m-1}-1\right)$$

is positive, then there exists a minimal code with parameters $[n, k]_{q^m/q}$.

For each $m, k \geq 2$ there exists a minimal $[2k+m-2, k]_{q^m/q}$ code.

Linearity index

U $[n,k]_{q^m/q}$ system

Linearity index:

$$I(U) := \max\{\dim_{\mathbb{F}_{q^m}}(H) : H \subseteq (\mathbb{F}_{q^m})^k, \mathbb{F}_{q^m} - \textbf{subspace}, H \subseteq U\}$$

Invariant for equivalent systems.
Related to the **generalized weights**

Linearity index of a code

C $[n, k]_{q^m/q}$ nondegenerate code and U an $[n, k]_{q^m/q}$ associated system.

$$I(U) = k - \min\{i : \ w_i(C) = n - (k - i)m\}$$

Linearity index and codes

C $[n, k]_{q^m/q}$ nondegenerate code and l is its linearity index.

$$w_{i+1}(C) - w_i(C) = m \Leftrightarrow i \geq k - l(C)$$

C $[n, k]_{q^m/q}$ nondegenerate code

$$l(C) \geq n - k(m - 1)$$

Linearity index and codes

U linear cutting blocking set with parameters $[n,k]_{q^m/q}$ and suppose there is $T \leq (\mathbb{F}_{q^m})^k$ a $\mathbb{F}_{q^m}$-subspace with $\dim_{\mathbb{F}_{q^m}}(T) = l$, $T \subseteq U$. Then U/T is isomorphic to a linear cutting blocking set of parameters $[n - lm, k - l]_{q^m/q}$.

U linear cutting blocking set with parameters $[n,k]_{q^m/q}$ with linearity index l. Suppose $k - l \geq 2$:

$$n - k \geq (l+1)(m-1)$$

Let C be the nondegenerate $[n,k]_{q^m/q}$ code associated to U. For each $1 \leq i \leq k - \lceil \frac{n-k+1}{m-1} \rceil - 1$, $w_i(C) > n - im$.

C nondegenerate $[(k-1)m, k]_{q^m/q}$ code with $l(C) = l$.

TFAE

- C minimal
- $l < k - 2$
- $w_2(C) > m$

$k \geq 3$ integer. There is a nondegenerate $[(k-1)m, k]_{q^m/q}$ code iff $m \geq 3$

Thank you for your attention!